Heredity, differential inclusions and viability in Coverings theory

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Abstract

I complete here the basis of the theory exposed in my first article [2] by some complementary results on stability properties between a set-valued map and its *characteristic boundary set-valued maps*, or its selections. I deduce then of these properties the main "Cauchy-Lipchitz theorems for differential inclusions and finally take up Viability Theory in Banach spaces.

Key-words

set-valued map; covering; selection; differential inclusion; viability

Contents :

Introduction

In my first article: "Set-Valued Analysis by Coverings" [2], I defined the notion of *covering* and showed its power to extend to the framework of set-valued maps each of the main theories of Functional Analysis : Algebra, Differential Calculus, Measure and Integration, Distributions and Fourier Analysis, by preserving their properties and main theorems. Coverings Theory is based on Lebesgue's spaces theory, differential geometry principles and, on the other hand, on fundamental notions and results of Casting and Aumann in the classical Set-valued maps Measure and Integration theory. Coverings Theory is complementary of Graphical Theory developed since Kuratowski and Bouligand exactly as Functional Analysis is complementary of Functions Graphical Theory.
The first article was dedicated to fundamental basis and for that contains very few applications. I complete here this work by some results more in that sense:

In section 1, I give first a property of heredity between a set-valued map and its *characteristic boundary set-valued maps.* Typical of context of Set-valued maps Theory, this result will bring us to examine differences between Coverings Theory itself and its use as tool of the of Set-valued maps Theory. Then I expose two theorems of existence and heredity of selections of a set-valued map.

In sections 2, I deduce of these results the main Cauchy-Lipschitz's theorems for differential inclusions and finally, in section 3, I take up Viability Theory, firstly in $\mathbb{R}^n$ with an adapted version of the fundamental Haddad's Theorem to the coverings context, then with a second theorem in general Banach spaces, more specific of coverings and ensuing of the Equilibrium Theorem.

Framework

Every notion used in this paper refers to definitions, fundamental results and notations exposed in [2]; we just recall that, for functions, all these definitions are equivalent to those of classical Functional Analysis [4].

If notions of classical Graphical Theory ([1], [3]) are occasionally used it will be specified as in [2] : Borel-measurability, Aumann-integrability and so on.

Notations

O open set of a Banach space E. $I = [0, +\infty[$. $F : I \times O \multimap E$ a set-valued map

$f : (t, y) = z \in I \times O \to f(z| \in L(b) = L^1(b, d\mu; E)$ an internal dynamic of F of basis $b \equiv (b, \mu) \equiv (n, b, \mu)$

$\underline{f} : z \to \underline{f}(z) = f(z|b)$ the subjacent set-valued map to the dynamic f

$f_J = (f_j)_{j \in J}$ a F-covering, i.e. a set of internal dynamics of F such as

$$\underline{f_J}(z) = \bigcup_{j \in J} \underline{f_j}(z) = F(z), \forall z.$$

Furthermore we just recall here that, following [2]:

1. a set-valued F is said to have analytic properties $P_1, ..., P_k$ when it admits a covering f_J which has the properties $P_1, ..., P_k$, i.e. is such as, for each j in J, f_j has the properties $P_1, ..., P_k$. In this case only such coverings are then considered.

2. A dynamic f of basis b is said 0-regular if, for each (t,y), $f(t,y|$ is continuous (and then uniformly continuous (see [2])) on b. In this case, its derivative, if it exists, is defined as an element of $C^0(b,L(I\times E,E)$, space included in $L(I\times E,C^0(b,E))$ and isomorphic to it if $dim(E)<\infty$.
A differentiable covering is then always 0-regular.

3. a set-valued map of finite order and convex images is always of order 1.

Remark: Norms in E and $\mathbb{R}^n$ $(n>1)$ are denoted $\|\|$, the uniform norm $\|\|_\infty$ and that of (b) $||_b$.

1 Heredity

1.1 Heredity and characteristic boundary set-valued map

1.1.1 Definitions

a. Let K a closed convex set of E, $\delta K=K-\overset{\circ}{K}$ its boundary. We call *characteristic boundary set* and denote $\delta_c K$ any subset of δK which closed convex-cover is K.

b. Let $F:I\times O\multimap E$ a set-valued map with closed convex images. We call *characteristic boundary set-valued map of* F and denote $\delta_c F$ any set-valued map from $I\times O$ in E which image, for each z in $I\times O$, is a characteristic boundary set of $F(z)$

Remark: If K is not bounded, the set of its characteristic boundaries can be empty and, obviously, a set-valued map may not admit characteristic boundary set-valued map.

1.1.2 Theorem

Let $z\multimap F(z)$ any union of set-valued maps with closed convex images $z\multimap F_i(z)\,,\,i\in I$. If for each i in I, F_i admits a characteristic set-valued map $z\multimap \delta_c F_i(z)$ measurable, respectively (locally) integrable, of C^k-class $k\leq\infty$, so is $z\multimap F(z)$.
Proof:

It follows directly from basis exposed in [2] considering the lemma:

lemma

Let a fixed i in I and $(z\rightarrow f_j(z|)_{j\in J}$ a covering of $z\multimap \delta_c F_i(z)$, then $(z\rightarrow f_{j,j'}(z|)_{(j,j')\in J^2}$ defined by:

$$f_{j,j'}(z|x,x',\lambda)=\lambda f_j(z|x)+(1-\lambda)f_{j'}(z|x')\,,\;\forall(x,x',\lambda)\in B_{j,j'}=b_j\times b_{j'}\times[0\,;1]$$

is a covering of $z\multimap F_i(z)$.

1.1.3 Application fields of coverings

This theorem is typical of set-valued maps point of view: the boundary determines the whole but it is not generally the same for coverings. Each result on coverings give a similar result on set-valued maps, nevertheless the passage of a covering to its subjacent set-valued map is almost always a loss of information and then often a loss of theoretical power and capacity of modeling. Let us give some basic examples:
Loss of theoretical power:

(1) A set-valued map can be on one hand continuous (i.e. admits a continuous covering)and on the other 0-regular (i.e. admits a 0-regular covering) but not both 0-regular and continuous (i.e. admits a 0-regular and continuous covering) [see Recalls or [2] for more details]

(2) If $f_J = (f_j, b_j)_{j\in J} : t \in I \to f_j(t| \ \in L(b_j, E)$ is a continuous covering, then it is locally integrable and its *primitive* $t \to \int_0^t f_J \, d\tau$ is a differentiable covering of derivative:

$$d\int_0^t f_J \, d\tau = f_J$$

.

It ensues that the continuous set valued map $\underline{f_J}$ is locally integrable and its *primitive* $t \to \underline{\int_0^t f_J \, d\tau}$ is a differentiable set-valued map but we have only

$$d\int_0^t \underline{f_J} d\tau \supset \underline{f_J}$$

The equality is become a simple inclusion (see [2]).

Loss of capacities of modeling:

(2) Let $R > 0, \omega > 0$ and consider the 4 following coverings defined on $\mathbb{R}$ and of subjacent images in $\mathbb{R}^2$ and let us suppose that their expressions are justified, for example, by a concrete observation:

f^1 of order 1, basis $b_1 = \left(\overline{B}_{\mathbb{R}^2}(0,1), {dx}/{\pi}\right) : \ f^1 : t \to f^1(t| \ : x \in b_1 \to R.x$

f^2 of order 1, basis $b_2 = \left([0,1] \times [0, 2\pi], {dx}/{\pi}\right) :$

$f^2 : t \to f^2(t| \ : x = (r,\theta) \to rR\left(\cos\left(\theta + 1_{\mathbb{Q}^*}(r)\,\omega t\right), \sin\left(\theta + 1_{\mathbb{Q}^*}(r)\,\omega t\right)\right)$

$f^3 = \left(f_j^3\right)_{j\in Q\cap[0,1]}$ of order $|\mathbb{N}|$, basis $b_{3,0} = b_2$, $b_{3,j>0} = \left([0, 2\pi], {dx}/{2\pi}\right)$

$f_0^3 : t \to f_0^3(t| \ : x = (r,\theta) \in b_{3,0} \to 1_{\mathbb{R}-\mathbb{Q}^*}(r)\, rR(\cos\theta, \sin\theta)$,

$f_{j>0}^3 : t \to f_j^3(t| \ : x = \theta \in b_{3,j} \to jR\left(\cos(\theta + \omega t), \sin(\theta + \omega t)\right)$

f^4 of order 1, basis $b_4 = b_2$:

$f^4 : t \to f^4(t| : \ x = (r,\theta) \in b_4 \to rR\left(\cos\left(\theta + (1-r)\,\omega t\right), \sin\left(\theta + (1-r)\,\omega t\right)\right)$

The 4 coverings have the same subjacent invariant set-valued map : $F : t - \circ F(t) = \overline{B}_R(0,R)$ and are of C^∞ class.

The first two coverings are actually equivalent, the difference being just a singularity (see [2]) and then the loss of information for F is null in case of f^1, negligible in case of f^2. It is not the same for f^3 and f^4 where the loss is complete: only coverings are able to model these cases, set-valued maps can not do that

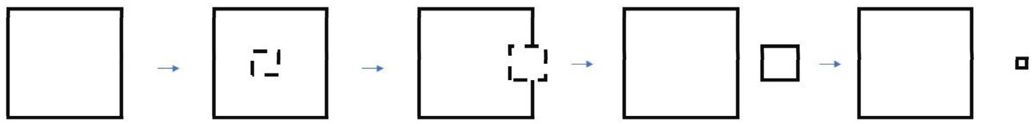
Furthermore the assumptions of the theorem show the flexibility of the theory; actually, neither convexity or neither connexity of the (subjacent) images nor global regularity of its boundary are required so that, for example, a covering or a set-valued map is differentiable or even of class C^k, $k \leq \infty$. *A simple example*:

Let $E = \mathbb{R}^2$, $c > 0$, and $(f_i)_{i=1,2}$ the following covering of order 2:

For $-1 < t < 9 \ : \ z \to (f_i(z|\,)_{i=1,2}$ of basis $(2, b, \mu)$,$x = (x_1, x_2) \in b = \left[-{1}/{2}; {1}/{2}\right]^2$, $d\mu = dx$

$f_1(z|\,x) = y + cx, \forall t \ ; \ \ f_2(z|x) = y + \left(\phi(t)\, c + \varphi(t)\, cx_1, \varphi(t)\, cx_2\right)$

with ϕ, φ Urysohn's functions such as: supp$\phi \subset [0, +\infty[$, $\phi_{|[1;10]} = 3$; supp$\varphi \subset [0; 6]$ $\varphi_{|[1;5]} = {1}/{4}$ defines a set-valued map which models the following plan:



In conclusion of these remarks I think that Coverings Theory is an efficient complementary tool for Set-Valued Analysis but furthermore has a real capacity to create models at the same time realistic and mathematically strong of complex phenomena as storms for example.

Remark: Existence and expressions of coverings of δF or $\delta_m F$ can be obtained by using differential geometry tools [2] and Theorem 1.1.2 gives then coverings of F without loss of analytic properties.

1.2 Heredity and selections

1.2.1 Definitions

Let $(t,y) \multimap F(t,y)$ a set-valued map of covering f_J.
A selection ϕ of F will be said a *f_J-associated selection* if:

$$\forall T > 0,\ \exists j \in J \,/\, \forall (t,y) \in [0,T] \times O \ :\ \ \phi(t,y) \in \overline{f_j(t,y)}$$

Their set will be noted $S(f_J)$ and $S(F)$that of selections of F

Remark:
If f_J is not time-dependent, for every j in J, $f_j(y|$ is obviously identified with $\widetilde{f_j}(t,y| = f_j(y|$. Then the set $S(f_J)$ obviously contains time-independent selections ϕ, i.e. such as $\exists j \in J \,/\, \phi(y) \in co\underline{f_j}(y), \forall y \in O$, but obviously not only.

1.2.2 Theorem of equiregularity

Let $(t,y) \multimap F(t,y)$ a continuous set-valued map and let us assume that F is locally 0-equiregular:
$\forall (t_0, y_0), \exists U \in V(t_0, y_0)$ such as the family $(f_j(t,y|, b_j) j \in J, (t,y) \in U$ is 0-(uniformly) regular, then:
for each (t,y), by any point z of $F(t,y)$ cross a continuous selection:

$$\overline{\{\phi(t,y)\,,\ \phi \in S(F)\}} = F(t,y)$$

Proof:
Let f_J a covering of F, we have then $\forall (t_0, y_0), z \in F(t_0, y_0), \exists j \in J, \exists x_0 \in b_j / \ z = f(t_0, y_0|x_0)$ and then:
continuity:

$$\forall \varepsilon > 0, \exists \eta_1 > 0, \eta_1' > 0 \,/\, |t - t_0| < \eta_1, \|y - y_0\| < \eta_1' \Rightarrow |f_j(t,y| - f_j(t_0, y_0|\,|_{b_j} < \varepsilon/3$$

local 0-equiregularity::

$$\begin{aligned}&\exists \eta_2 > 0, \eta_2' > 0 \,/\, \forall \varepsilon > 0, \exists \eta'' > 0 :\\ &\qquad |t - t_0| < \eta_2, \|y - y_0\| < \eta_2', \|x - x_0\| < \eta'' \Rightarrow \|f(t,y|x) - f(t,y|x_0)\| < \varepsilon/3\end{aligned}$$

Consider then the selection of F $t \to f(t, y|x_0)$, we have:

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \eta > 0, \eta' > 0 \ : \\
&\|f(t, y|x_0) - f(t_0, y_0|x_0)\| \leq \tfrac{1}{\mu_j(\{x \in b_j,\, \|x - x_0\| < \eta''\})} \int_{\{x \in b_j,\, \|x - x_0\| < \eta''\}} \|f(t, y|x_0) - f(t, y|\xi)\| \, d\mu_j(\xi) \\
&\qquad + |f(t, y| - f(t_0, y_0|\,|_{b_j} + \tfrac{1}{\mu_j(\{x \in b_j, \|x - x_0\| < \eta''\})} \int_{\{x \in b_j,\, \|x - x_0\| < \eta''\}} \|f(t_0, y_0|x_0) - f(t_0, y_0|\xi)\| \, d\mu_j(\xi) \\
&\quad \leq 2.\tfrac{\varepsilon}{3}.\tfrac{1}{\mu_j(\{x \in b_j, \|x - x_0\| < \eta''\})} \int_{\{x \in b_j,\, \|x - x_0\| < \eta''\}} d\mu_j(\xi). + \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

The result follows.

Remark

It is obvious that if F is *totally continuous*, i.e. admits a covering f_J such as:

$$(t, y, x) \in I \times O \times b_j \to f_j(t, y|x), \forall j$$

then F is locally 0-equiregular.

1.2.3 Theorem of barycentric selections

Let $(t, y) \multimap F(t, y)$ a set-valued map with convex closed images and $(t, y) \to f(t, y|$ an internal dynamic of F of basis (b, μ).
For any $\left(b', \frac{\mu}{\mu(b')}\right)$ sub-basis of b, $(t, y) \to f_{b'}(t, y) = \frac{1}{\mu(b')} \int_{b'} f(t, y|x) \, d\mu(x)$ is a selection of F measurable, respectively (locally) integrable, y-locally Lipschitz, of C^k-class ($k \leq \infty$) if F is. Such a selection will be said *barycentric f-selection of F.*

Furthermore, if F is 0-regular [2], for any (t, y), the set $S(F)$ of its selections is such as:

$$\overline{\{\phi(t, y)\,,\ \phi \in S(F)\}} = F(t, y)\,;$$

more precisely, for any covering f_J of F , the set $S(f_J)$ of f_J-associated selections of F is such as:

$$\overline{\{\phi(t, y)\,,\ \phi \in S(f_J)\}} = F(t, y)\,.$$

Proof

i. $\forall z,\ f(z| \in L(b)\ ,\ \mu(b') > 0$ (See [2]) : $f_{b'}(z)$ is defined and is a barycentric value of $f(z|x)$ on b'. As $F(z)$ is close and convex $f_{b'}(z)$ is in $F(z)$ and so $f_{b'}$ is a selection of F such as $f_b(z) \in \overline{co}\underline{f}(z) \subset F(z), \forall z.$

ii. If $z \to f(z|$ is measurable, it is the simple limit of measurable staged functions $z \to g^m(z| = \sum_i 1_{\Omega_i^m} g_i^m$ with $\Omega_i^m \subset I \times O$ measurable such as $\mu(\Omega_i \cap \Omega_{i'}) = 0$ for $i \neq i'$ and $g_i^m \in L(b)$. Then, for any z, $\forall m$, $\exists i_m \,/\, z \in \Omega_i^m$ and so $f(z| \ = \lim_m g_{i_m}^m$ in $L(b)$:

$$|f_{b'}(z) - c_m| \leq \frac{1}{\mu(b')} \left|f(z| - g_{i_m}^m\right|_{\,b} \underset{m \to \infty}{\to} 0 \text{ with } \mathrm{c_m} = \frac{\int_{\mathrm{b'}} \mathrm{g}_{\mathrm{i_m}}^{\mathrm{m}} \mathrm{d}\mu}{\mu(\mathrm{b'})}.$$

So f_b' is a simple limit of measurable staged functions and then is a measurable function.

iii. Assume that f is y-locally lipschitz:

$$\forall (t_0, y_0), \exists \delta, \varepsilon, k > 0 / \ |f(t, y_1| - f(t, y_2|\,|_b \leq k \|y_1 - y_2\| \text{ if } |t - t_0| < \delta, \|y_i - y_0\| < \varepsilon, i = 1, 2$$

Then:

$$\|f_{b'}(t, y_1) - f_{b'}(t, y_2)\| \leq \frac{1}{\mu(b')} \int_{b'} \|f(t, y_1| - f(t, y_2|\ \| \, d\mu \leq \frac{k}{\mu(b')} \|y_1 - y_2\|$$

iv. $\|f_{b'}(z)\| \leq \frac{1}{\mu(b')} |f(z| \,|_b$ and so f'_b is (locally) integrable, respectively continuous, if $z \to f(z|)$ is.

v. Let $z \to f(z|\ C^k$, $\infty > k > 1$. For $0 \leq j < k$, we have successively:

$$f^{(j)}(z+h| \ = f^{(j)}(z| \ + f^{(j+1)}(z| \ .h \ + \|h\| \ \varepsilon(z;h| \ / \lim_{\|h\|\to 0} |\varepsilon(z;h| \,|_b = 0$$

$$f_{b'}^{(j)}(z+h) \ = f_{b'}^{(j)}(z) \ + f_{b'}^{(j+1)}(z) \ .h \ + \|h\| \ \varepsilon_{b'}(z;h) \ / \lim_{\|h\|\to 0} \|\varepsilon_{b'}(z;h)\| \leq \frac{1}{\mu(b')} \lim_{\|h\|\to 0} |\varepsilon(z;h| \,|_b = 0$$

So $f_{b'}^{(j)}$ is differentiable and $f_{b'}^{(j+1)}(z) = \left(f_{b'}^{(j)}\right)'(z) = ... = f_{b'}^{(j+1)}(z)$

vi. Let F 0-regular.
$\forall \omega_0 \in F(z)$, $\exists f$ internal dynamic of F, $\exists x_0 \in b$ such as $f(z|$ continue on b and $\omega_0 = f(z|x_0)$.
Let: $\varepsilon > 0$ and $b' = \{x \in b / \ \|x - x_0\| < r \ / \ \|f(z|x) - f(z|x_0)\| < \varepsilon$. We obtain then:
$\|f_{b'}(z_0) - \omega_0\| \leq \frac{1}{\mu(b')} \int_{b'} \|f(z_0|x) - \omega_0\| d\mu(x) \leq \varepsilon$ and the result follows.

Corollary:

Let $(t,y) \multimap F(t,y)$ a set-valued map of order at most countable with convex closed images. There exists a sequence $(\phi_k)_{k\in\mathbb{N}}$ of measurable, respectively (locally) integrable, y-locally Lipschitz, of C^k-class ($k \leq \infty$) selections of F such as $\forall (t,y) \in I \times O$, $F(t,y) = \overline{\underset{k\geq 0}{\cup} \varphi_k(t,y)}$ if F is.
Proof:
Let f_J a covering of order at most countable of F.
For each j in J it exists a dense sequence (x_k^j) in b_j and then the sequence of barycentric selections $\phi_{j,k,m}$ associated to $\left\{f_j, b_{j,k,m} = \overline{B}\left(x_k^j, {}^1\!/_m\right) \cap b_j \,/\, j \in J, k \geq 0, m > 0\right\}$ answers to the question

Comments:

(1) This theorem and its corollary are in moved closer to the Characterization Theorem of Borel-Measurability ([1], [3]) and the problem of eventual cases of reciprocity other than measurability is open.

(2)In cases of measurability and (local) integration, selections and closed relations with Borel-measurability and Aumann-integration have already been exposed in [2].

2 Cauchy-Lipschitz Theorems for differential inclusions

2.1 The fundamental theorem:

Let $(t,y) \in I \times O \multimap F(t,y) \subset E$ a continuous set-valued map with closed convex images, y_0 in O, and let us consider the differential inclusion:

$$(I) \quad y'(t) \in F(t, y(t)), \ \ y(0) = y_0$$

Assume that F is locally Lipschitz in y (See [2]), then:

$$\exists, \ 0 < T \leq \infty / \exists y : [0, T[\to E, \text{ of C}^1\text{−class solution of (I) for } 0 \leq \text{t} < \text{T}$$

Furthermore, let $S_F(y_0)$ the set of solutions of (I), if F is at least 0-regular, we have:

$$\overline{\{y'(0) \ / \, y \in S_F(y_0)\}} = F(0, y_0)$$

More precisely, let f_J any covering of F and $S_{f_J}(y_0)$ the set of solutions deduced of the set $S(f_J)$ of its associated selections, we have then:

$$\overline{\{y'(0)\ /\ y \in S_{f_J}(y_0)\}} = F(0, y_0)$$

Proof:

Following theorem 1.2.2, that ensues directly from the classical fundamental Cauchy-Lipschitz theorem for functions applied in barycentric selection.

2.2 C^k and parametric first order differential inclusions:

Theorem:

Let Λ an open set of $\mathbb{R}^p$, $(t, y\,; \lambda) \in I \times O \times \Lambda \multimap F(t, y\,; \lambda) \subset E$ a set-valued map with closed convex images, y_0 in O, and let us consider the differential inclusion of parameter λ:

$$(I_\lambda) \quad y'(t) \in F(t, y(t)\ ; \lambda)\,, \ \ y(0) = y_0$$

Assume that:

$$\text{for each } \lambda \text{ in } \Lambda, (\mathrm{t}, \mathrm{y}) \multimap \mathrm{F}(\mathrm{t}, \mathrm{y}\,; \lambda) \ \text{ is of } \mathrm{C}^{\mathrm{k}}-\text{class}, \ 0 < \mathrm{k} \leq +\infty$$

Then:

$$\forall \lambda \in \Lambda, \exists T_\lambda > 0, \exists y_\lambda : [0, T_\lambda[\to E \ C^{k+1} \text{solution of } \ (\mathrm{I}) \ \forall 0 \leq \mathrm{t} < \mathrm{T}_\lambda$$

Furthermore:

if $(t, y\,; \lambda) \multimap F(t, y\,; \lambda)$ is of C^{h} – class, $0 \leq \mathrm{h} \leq \mathrm{k}$, $(\mathrm{t}, \lambda) \to \mathrm{y}(\mathrm{t}, , \lambda) = \mathrm{y}_\lambda(\mathrm{t})$ is of C^k-class.

Proof:

If E and O are respectively replaced by $E \times \mathbb{R}^p$ and $\mathrm{O} \times \Lambda$ in 1.1.2, we obtain:

If $(\mathrm{t}, \mathrm{y}\,; \lambda) \to \mathrm{f}(\mathrm{t}, \mathrm{y}\,; \lambda|$ is of C^{k} – class, $0 \leq \mathrm{k} \leq +\infty$, so are its barycentric selections.

The theorem follows then from the classical Cauchy-Lipschitz theorem applied in barycentric selections of $(t, y\,; \lambda) \multimap F(t, y\,; \lambda)$.

2.3 Differential inclusion of m order

Theorem:

Let $(t, y, z_1, ..., z_{m-1}) \in I \times O^{1+m'} \multimap F(t, y, z_1, ..., z_{m-1}) \subset E$a set-valued map with closed convex images, y_0,y_1, ..., y_{m-1} m arbitrary points in O and let us consider the differential inclusion of m order:

$$(I) \quad y^{(m)}(t) \in F\left(t, y(t), ..., y^{(m-1)}(t)\ ; \lambda\right); \ \ y^{(i)}(0) = y_i, i = 1, ..., m-1$$

Assume that:

$$(t, y, z_1, ..., z_{m-1}) \multimap F(t, y, z_1, ..., z_{m-1}) \ \text{ is of } \mathrm{C}^{\mathrm{k}} - \text{class}, 1 \leq \mathrm{k} \leq \infty$$

Then:

$$\exists T > 0 / \ \exists y : [0, T[\to E\,, \ C^{k+m}, \text{solution of } \ (\mathrm{I}) \ \text{ for } 0 \leq \mathrm{t} < \mathrm{T}.$$

Proof:

To (I) we associate the differential inclusion :

$$Y' \in \tilde{F}(t,Y), \quad Y_0 = (y_0,...,y_{m-1}),$$

where $(t,y,z_1,...,z_{m-1}) \multimap \tilde{F}(t,y,z_1,...,z_{m-1}) = \{z_1\} \times ... \times \{z_{m-1}\} \times F(t,y,z_1,...,z_{m-1}) \subset E^m$ is of C^k-class if F is.
Theorem 2.3 gives then:

$$\exists T>0,\ \exists Y:[0,T[\to E^m,\ C^k\,/\,Y'(t)\in\tilde{F}(t,Y(t))\ \text{for},\ 0\leq \mathrm{t} < \mathrm{T}\ \text{and}\ \mathrm{Y}(0)=(\mathrm{y}_0,...,\mathrm{y}_{\mathrm{m}-1})$$

i.e. with $Y=(y,z_1,...,z_{m-1}):\ y'=z_1,\ z_1'=z_2,\ ...\ ,\ z_{m-1}'\in F(t,y,z_1,...,z_{m-1})$.
The result follows.

3 Viability

3.1 Haddad's Theorem

This first theorem, throw a lemma, replaces the set-valued map in the classical context of Graphical Theory and is then just an adaptation of the fundamental Haddad's Theorem:
Theorem:
Let $F:\mathbb{R}^n - \circ\mathbb{R}^n$ a continuous set-valued map of finite order with convex images and K a locally compact subset of $\mathbb{R}^n$.
Assume that F is locally 0-equiregular then K enjoys the local viability property if and only if K is a viability domain of F.

Lemma:
A set-valued map of finite order continuous and locally 0-equiregular is Kuratowski upper semicontinuous.
Proof:
$(f_j)k=1,...,kJ$ is continuous and uniformly 0-equiregular: $\forall\varepsilon>0, \exists\eta>0, \exists\eta'>0$ if
$|y-y_0|<\eta, |x-x_0|<\eta', \forall x_0\in b_j$ we have:

$$|f_j(y|-f_j(y_0|\,|_{b_j} < \varepsilon/3 \quad rmand \quad \|f(y|x_0)-f(y|x_0)\| < \forall j=1,...k$$

Let z in $F(y)$, it exists $j=1,...,k$ and x_0 in b_j such as $f_j(y|x_0)=z$. Let then $z_0=f(y_0|x_0)$, we have $z_0\in F(y_0)$ and

$$\begin{aligned}\|z-z_0\| = \|f(y|x_0)-f(y_0|x_0)\| &\leq \tfrac{1}{\mu_j(\{x\in b_j,\,\|x-x_0\|<\eta'\})}\int_{\{x\in b_j,\,\|x-x_0\|<\eta'\}}\|f(y|x_0)-f(y|\xi)\|\,d\mu_j(\xi)\\ &+|f(y|-f(y_0|\,|_{b_j}+\tfrac{1}{\mu_j(\{x\in b_j,\|x-x_0\|<\eta'\})}\int_{\{x\in b_j,\,\|x-x_0\|<\eta'\}}\|f(y_0|x_0)-f(y_0|\xi)\|\,d\mu_j(\xi)\\ &\leq 2.\tfrac{\varepsilon}{3}.\tfrac{1}{\mu_j(\{x\in b_j,\|x-x_0\|<\eta'\})}\int_{\{x\in b_j,\,\|x-x_0\|<\eta'\}}d\mu_j(\xi).+\tfrac{\varepsilon}{3}=\varepsilon\end{aligned}$$

Hence: $\forall\varepsilon>0, \exists\eta>0,\ \|y-y_0\|<\eta\Rightarrow F(y)\subset\overline{B}(F(y_0,\varepsilon))$ and tha achieves the proof of the lemma

Proof of theorem 3.1:
F is then Kuratowski upper semicontinuous and furthermore has compact images:
Let $f_j,(b_j,\mu_j), j=1,...,k$ a covering of F and $y_0\in O$, we have:
$\forall y,\ F(y)=\underset{j=1,...,k}{\cup}\underline{f_j}(y)=\underset{j=1,...,k}{\cup}\underline{f_j}(y|b_j)$ which is compact because f_j is 0-regular and then $f_j(y|b_j)$ is the continuous image of a compact.
Hence all hypotheses of Haddad's Theorem are verified and the result follows.

Comments: More generally the result of this lemma gives obviously an access to the numerous theorems of Graphical Theory and Viability Theory based on Kuratowski upper semicontinuity.

3.2 Viability theorem in Banach spaces

3.2.1 Preliminaries

(1) Let $0 < T \leq \infty$, we denote X_T the space $C^0\left(\overline{[0,T[}^{\mathbb{R}_+}, E\right)$ with its usual topology of compact convergence; more precisely, for $T < \infty$, X_T is then the Banach space $\left(C^0\left([0,T], E\right), \|\|_\infty\right)$ and for $T < \infty$ and the Frechet space $C^0\left([0,\infty[, E\right)$ endowed with the topology of locally uniform convergence.

Let K a convex compact of E.
The closed cover of any convex bounded subset of $C^1(\overline{[0,T[}^{\mathbb{R}_+}, K)$ endowed with the C^1 compact convergence topology is, following the Ascoli's Theorem, a compact of X_T: It is indeed the closed cover of a set of equicontinuous functions on $\overline{[0,T[}^{\mathbb{R}_+}$ which set of values on each t_0 is included in K and so relatively compact.
More generally, we will denote $\tilde{K}_T$ any convex compact set of elements of X_T viable in K.

(2) Let $F : \mathbb{R}_+ \times O - \circ E$ a set-valued map of covering $f_J = (f_j, b_j, \mu_j)_{j\in J}$ and $\alpha > 0$.
For each j in J, we defined formally the set-valued map $\tilde{F}_j$ by:

$$\widetilde{F_j} : y() \in X_T - \circ \left\{ t \to y_j^{x,r}(t) = \int_0^t f_j^{x,r}(\tau, y(\tau)) .d\tau \,,\; (x,r) \in b_j(\alpha) \right\}$$

where: $\quad b_j(\alpha) = \left\{(x,r) \in b_j \times [0,1] \;/\; \overline{B}(x,r) \subset b_j\right\}, \qquad \mu_{j,r} = \mu_j\left(\overline{B}(x,r)\right)$

$$f_j^{x,r}(t,y) = \frac{1}{\mu_{j,r}} \int_{\overline{B}(x,r)} f_j(t, y| \, d\mu_j$$

and the set-valued map $\tilde{F}$ by:

$$\tilde{F} : y() \in X_T - \circ \, \overline{co} \bigcup_{j\in J} \tilde{F}_j(y())$$

3.2.2 Theorem:

Let $F : \mathbb{R}_+ \times O - \circ E$ a set-valued map with closed convex images of covering $f_J = (f_j, b_j, \mu_j)_{j\in J}$ and $\alpha > 0$.
We assume that F is locally strongly continuous i.e. the covering f_J is a locally equicontinuous family.
Let K a convex compact of E and $y_0 \in K$:

If it exists $0 < T \leq \infty$, $\alpha > 0$ and $\tilde{K}_T$ such as the set-valued map $y_0 + \tilde{F}$ is either inward or outward on $\tilde{K}$, then it exists $y() \in X_T$ of C^1 class, viable in K, and such as:

$$y(0) = y_0, \qquad y'(t) \in F(t, y(t)), \forall t \qquad (I)$$

Proof:
Step 1: $y() - \circ \, \widetilde{F}(y())$ is defined on X_T with (closed convex) images in X_T:
$\forall y() \in X_I, \forall j \in J, \forall (x,r) \in b_j(\alpha), \; y_j^{x,r}()$ is defined and continuous (actually C^1) on $\overline{[0,T[}^{\mathbb{R}_+}$ because $f_j^{x,r}$ is a barycentric selection of a continuous set-valued map with convex closed images and then is continuous, hence so is $\tau \to f_j^{x,r}(\tau, y(\tau))$.

Step 2: $\tilde{F}$ is hemicontinuous on X_T:

Let $y_k() \to y_\infty()$ and $u \in X_T'$ dual space of X_T.

(1) Let assume first that $T < \infty$:

We have: $\forall k, \forall \varepsilon > 0, \exists z_k\left(\right) \in \tilde{F}\left(y_k\left(\right)\right) : \sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq < u, z_k\left(\right) > + \varepsilon/3$

Then: $\exists z'_k\left(\right) \in co\left(\underset{j\in J}{\cup}\tilde{F}_j\left(y_k\left(\right)\right)\right) : \left\|z_k\left(\right) - z'_k\left(\right)\right\|_\infty \leq \eta$ such as $|< u, z_k\left(\right) - z'_k\left(\right) >| \leq \varepsilon/3$

and then: $\sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq < u, z'_k\left(\right) > + 2\varepsilon/3$

$z'_k\left(\right) = \sum_{i=1}^{m}\lambda_i z^i\left(\right)$, with $\lambda_i > 0, \forall i = 1, ..., m$, and $\sum_{i=1}^{m}\lambda_i = 1$, Hence

$< u, z'_k\left(\right) > = \sum_{i=1}^{m}\lambda_i < u, z^i\left(\right) > \leq \max_{i=1,\dots,m} < u, z^i\left(\right) >$ and then:

$\forall k, \forall \varepsilon > 0, \exists j_k \in J, \exists z^{j_k} \in \tilde{F}_j\left(y_k\left(\right)\right) : \ \sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq < u, z^{j_k}\left(\right) > + 2\varepsilon/3$

We have: $z^{j_k}\left(t\right) = \int_0^t f_{j_k}^{x_k, r_k}\left(\tau, y_k\left(\tau\right)\right)d\tau$ for a suitable $\left(x_k, r_k\right) \in b_{j_k}\left(\alpha\right)$

Let us consider $z_{j_k}\left(\right) \in \tilde{F}_{j_k}\left(y_\infty\left(\right)\right)$ defined by: $z_{j_k}\left(t\right) = \int_0^t f_{j_k}^{x_k, r_k}\left(\tau, y_\infty\left(\tau\right)\right)d\tau$,
we have:

$$\begin{aligned}\left\|z^{j_k}\left(t\right) - z_{j_k}\left(t\right)\right\| &\leq \int_0^t \frac{1}{\mu_{j_k, r_k}}\int_{\overline{B}\left(x_k, r_k\right)}\left\|f_{j_k}(\tau, y_k\left(\tau\right)|-f_{j_k}(\tau, y_\infty\left(\tau\right)|\right\| d\mu_{j_k} d\tau \\ &\leq \frac{1}{\alpha}\int_0^t \left|f_{j_k}(\tau, y_k\left(\tau\right)|-f_{j_k}(\tau, y_\infty\left(\tau\right)|\right|_{b_{j_k}} d\tau\end{aligned}$$

Hence, following (2): $\forall k \geq k_\varepsilon$ such as $\left\|y_k\left(\right) - y_\infty\left(\right)\right\|_\infty \leq \eta' \leq \eta$

$$\left|f_{j_k}(\tau, y_k\left(\tau\right)|-f_{j_k}(\tau, y_\infty\left(\tau\right)|\right|_{b_{j_k}} \leq \frac{\varepsilon\alpha}{3T}, \forall j \in J, \forall \tau \in \left[0, T\right].$$

We obtain then: $\left\|z^{j_k}\left(\right) - z_{j_k}\left(\right)\right\|_\infty \leq \frac{1}{\alpha}\int_0^t \frac{\varepsilon\alpha}{3T}d\tau \leq \varepsilon/3$ and finally:

$$\begin{aligned}&\forall \varepsilon > 0, \exists j_k \in J, \exists z_{j_k} \in \tilde{F}_{j_k}\left(y_\infty\left(\right)\right) \subset \tilde{F}\left(y_\infty\left(\right)\right), \exists k_\varepsilon : \\ &\quad \forall k \geq k_\varepsilon, \sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq < u, z_{j_k}\left(\right) > + < u, z^{j_k}\left(\right) - z_{j_k}\left(\right) > + 2\varepsilon/3 \leq < u, z_{j_k}\left(\right) > + \varepsilon\end{aligned}$$

Hence: $\forall \varepsilon > 0, \ \exists k_\varepsilon : \forall k \geq k_\varepsilon, \sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq \sigma\left(\tilde{F}\left(y_\infty\left(\right)\right), u\right) + \varepsilon$:

and then: $\limsup_{n\to\infty} \sigma\left(\tilde{F}\left(y_k\left(\right)\right), u\right) \leq \sigma\left(\tilde{F}\left(y_\infty\left(\right)\right), u\right)$which achieves the proof of step 2.

(2) Let assume now that $T = \infty$, the same preceding proof applied to $T' < \infty$ establishes then the validity of the limit inequality on any convex compact vicinity of $[0, \infty[$and then proofs the searched hemicontinuity inequality.

Step 3: final proof of theorem.

The theorem ensues then directly of the classical Equilibrium Theorem applied in the set-valued map $\tilde{F} + y_0$ for suitable values of T, α and choice of compact $\tilde{K}$.

Comments:

(1) If the set-valued map F is strongly locally bounded,i.e. the covering f_J is a locally equibounded, which is equivalent to locally bounded in sense of the Graphical Theory, any $y()$ viable in K and solution of the differential inclusion (I) is element of a set $\tilde{K}$ of C^1 bounded type. (2) Any set-valued map continue and of finite order is obviously strongly continue.

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