On the non homogeneous incompressible Navier-Stokes equation.

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***Abstract*:** For the main, we first adapt a classical result of existence of a maximal local-in-time solution to the homogeneous incompressible Navier-Stokes equation to our particular framework in a slightly stronger form, and next prove the globality-in-time of the obtained solution and establish a kind of optimality of this result

***Keywords*:** Non linear analysis; Navier-Stokes equation; Heat equation; Schwartz spaces; Soblolev spaces; Leray's projector

Introduction

As said in the Abstract, we first recall a main and classical result from the reference book [6] *Vorticity and Incompressible Flow* , §3: *Energy Methods* from A.J Majda and A.L Bertozzi , that we will both slightly strengthen and enlarge (non homogeneous equation) on one hand and, on the other weaken by a lost of generality,our framework being more restrictive: constraint in the Schwartz functionnal space and vorticity in the kernel-set $H^{\infty,\, f} = \bigcap\limits_{m \geqslant 0} H^{m,\, f}$ of Sobolev spaces The proof of this new form of our starting result is closely similar for a very large part to the proof of the original result established in [6] p 96 to 112. Likewise, that of the kinetic energy inequality, necessary in this work, in its processes, is similar to that of the usual kinetic energy equality [2] p 5. So, we will not give detailed proofs of these results, but only reference proof elements step by step and leave to the reader an eventual completed detailed re-writing.

In a second part, we will aboard the proof that the obtained solution is global-in-time using processes based, on the one hand, on Hilbert Theory and Heat Equation Theory, on the other, on a original method of *break and rebuilt* as we will see then.

Finally, in a third part, we will complete this work determining the stability domain of the Leray projector in the space of Schwartz functions and, so, obtain a kind of optimality of the preceding result.

Notations:

Let $N \in \mathbb{N}$, $N \geqslant 3$

1. Spatial derivatives ∂^{β}, $\beta \in \mathbb{N}^N$ are a priori in the distribution sense and the time derivative ∂_t always is in the Fréchet sense. (Notation from [2])

2. Ω an open subset in $\mathbb{R}^m$: $C^k(\Omega) = C^k\left(\Omega, \mathbb{R}^N\right)$, $0 \leqslant k \leqslant +\infty$
For $m = N$, $\Omega = \mathbb{R}^N$: $C^k(\Omega)$ will be denoted C^k

3. *Lebesgue spaces* [7], [8] :

$$L^p = L^p\left(\mathbb{R}^N, \mathbb{R}^N\right),\ 1 \leqslant p \leqslant \infty, \text{ norm: } |w|_{L^p}$$

4. *Schwartz spaces* [8] :

$$S = S\left(\mathbb{R}^N, \mathbb{R}^N\right)\ ,\ \ S^f = \{w \in S / div\, w = 0\}$$

semi-norms systems: (1) $|w|^{i,m} = \sup_{|\beta|\leqslant m} \left|\left(1+|x|^2\right)^i \partial^\beta w(x)\right|_{L^\infty}, \quad i,m \geqslant 0$

(2) $|w|^{(m)} = \sup_{\max\{|\alpha|,|\beta|\}\leqslant m} \left|x^\alpha \partial^\beta w\right|_{L^\infty}, m \geqslant 0$

$S(0,T^*) = S\left([0,T^*[\times \mathbb{R}^N, \mathbb{R}^N\right);\ 0 < T^* \leqslant +\infty$

semi-norms system: $||w||^{i,m} = \sup_{j+|\beta|\leqslant m} \left|\left(1+t+|x|^2\right)^i \partial_t^j \partial^\beta w(x)\right|_{L^\infty}$

5. *Sobolev spaces* [6], [8] and *Leray's projector* [2], [6]:

$H^m = H^m\left(\mathbb{R}^N, \mathbb{R}^N\right)$, norm: $\|\|_m,\ m \geqslant 0$

For all $m > N/2$, H^m is a Banach algebra and $\|v.w\|_m \leqslant c. \|v\|_m . \|w\|_m$

Hodge theorem:

$$P : w \in H^m \to Pw \in H^m \Big/ \left| \begin{array}{l} div\,(Pw) = 0 \\ w = Pw + \nabla\phi \end{array} \right. ;\quad ||Pw||_m \leqslant ||w||_m,\ H^{m,f} = PH^m$$

Fourier expression of P :

$Pw = (l.\widehat{w})^{\vee}$ with $l_j{}^k(x) = \left(\delta_j{}^k - \frac{x_j x^k}{|x|^2}\right)$ [2]

where $w \to \widehat{w}$ and $w \to w^{\vee}$ are the Fourier and co-Fourier transforms [1], [8]

I. Local-in-time Existence Theorem:

We define first the followings sets:

$$H^{\infty,f} = \bigcap_{m\geqslant 0} H^{m,f} \ \ \&\ \ H^{m\,f} = \{w \in H^m \,/\, div\, w = 0\} \,/\ \ 0 \leqslant m \leqslant +\infty$$

$$C^k\left([0,T^*[\,, H^\infty\right) = \bigcap_{m\geqslant 0} C^k\left([0,T^*[\,, H^m\right)\ \ /\ \ 0 \leqslant k \leqslant +\infty,\ \ 0 < T^* \leqslant +\infty$$

Our reference results are the following:

[2] p 5: *I.3.2 Energy equality* : if $u_0 \in H^{0,f}$ and $u \in C^1([0,T*), H^{0,f})$ are such as: $u(0,\ ) = 0,\quad \partial_t u - \nu\,\Delta u + (u.\nabla u) = 0$ then u verifies the Energy equality:

$$\forall t \in [0,T*)\,,\ \ 1/2\,|u(t,\ )|^2_{L^2} + \int_0^t |\nabla u(s,\ )|^2_{L^2} ds = 1/2\,|u_0|^2_{L^2}$$

[6] *Theorem 3.4* p 104. / *Corollary* 3.2 p 112: Given an initial condition $u_0 \in H^{m,f}$, $m \geqslant \left[N/2\right] + 2$, then for any viscosity $\nu > 0$, there exists a maximal time of existence T^* (possibly infinite) and a unique solution $u \in C^0([0,T^*), H^{m,f}) \cap C^1([0,T^*), H^{m-2,f})$ to the Navier-Stokes equation $u(0,\ ) = 0,\quad \partial_t u - \nu\,\Delta u + (u.\nabla u) = 0$.

We have to note that in this result T^* depends from m and that is the core of the problem to be solve.

We will modify these results as it follows:

Theorem I.1: *Energy inequalities*

Let $0 < T^* \leqslant +\infty$, $0 < m \leqslant +\infty$, $u_0 \in H^{m,f}$, $f \in S(0,+\infty)$, $\nu > 0$, and $u \in C^1\left([0,T^*[\,,\, H^{m,f}\right)$ such that:

$$\partial_t u - \nu\,\Delta u + P\,(u.\nabla u) = Pf$$

then u verifies the energy inequalities:

$$1/2\,|u\,(t,)|_{L^2}^2 + \nu \int_0^t |\nabla u\,(s,)|_{L^2}^2 ds \leqslant \int_0^t |u\,(s,)|_{L^2}\,.\,|Pf\,(s,)|_{L^2}\,ds + 1/2\,|u_0|_{L^2}^2 \quad (Eq.I.1)$$

$$\sup_{0\leqslant t<T^*} |u\,(t,)|_{L^2} \leqslant 2\int_{s\geqslant 0} |Pf\,(s,)|_{L^2}\,ds + |u_0|_{L^2} = E_{(u_0,f)} \qquad (Eq.I.2)$$

reference proof elements:

$(Eq.I.1)$: [2] p 5.
$(Eq.I.2)$:
$t \to u\,(t,)$ is continuous from $[0,T^*[$ to L^2, and $\int_{s\geqslant 0} |Pf\,(s,)|_{L^2}\,ds < \infty$ if $f \in S(0,+\infty)$, see [8], the processes used in [2] give us immediately:
$\forall 0 < t < T^*$, $(\sup_{s\in[0,t]} |u\,(s,)|_{L^2})^2 - 2\left(\int_{s\geqslant 0} |Pf\,(s,)|_{L^2}\,ds\right)(\sup_{s\in[0,t]} |u\,(s,)|_{L^2}) - |u_0|_{L^2}^2 \leqslant 0$
and the result follows.

Theorem I.2 : *Local-in-time existence* For any initial velocity $u_0 \in H^{\infty,f}$ and external force $f \in S(0,+\infty)$, there exists one maximal interval $[0,T^*[$, $0 < T^* \leqslant +\infty$ and one and only one $u \in C^1\left([0,T^*[, H^{\infty,f}\right)$ such that:

$$\left|\begin{array}{ll} (i) & \forall t \in [0,T^*[\,,\ div\,u\,(t,) = 0 \\ (ii) & u\,(0,) = u_0 \\ (iii) & \sup_{t\in[0,T^*[} |u\,(t,)|_{L^2} < \infty \\ (iv) & \partial_t u - \nu\,\Delta u + P\,(u.\nabla u) = Pf \end{array}\right.$$

Furthermore, we have then:

$$\sup_{t\in[0,T^*[} |u\,(t,)|_{L^2} \leqslant E_{(u_0,f)} = 2\int_{s\geqslant 0} |f\,(s,)|_{L^2}\,ds + |u_0|_{L^2} < \infty$$

$$\forall t \in [0,\,T^*[\,,\ \ u\,(t,\,) \in C_0^\infty = \left\{ w \in C^\infty \,/\, \lim_{|x|\to+\infty} \partial^\alpha u\,(t,x) = 0,\ \forall\alpha \right\}$$

reference proof elements: [6] p 100 - 112:

first step: the regularized equation:

The regularized equation considered here is time-dependent (second member $f(t,\ )$), so, the Picard theorem is inadequate and has to be replaced by the Cauchy-Lipschitz theorem; that does

not change the structure of the proof. The upper bound $\sup_{0\leqslant t\leqslant T}|v^{\varepsilon}|_{L^2}\leqslant E(u_0,f)$ obviously replaces $\sup_{0\leqslant t\leqslant T}|v^{\varepsilon}|_{L^2}\leqslant |u_0|_{L^2}$ in ref. Eq (3.53).
Similarly to the reference, given that f belongs to $S(0,\infty)$, see [8], we have the bound:

$$\frac{d}{dt}\|u^{\varepsilon,m}(t,\backslash)\|_m\leqslant c(E(u_0,f),\varepsilon)\|u^{\varepsilon,m}\|_m+\|f(t,\backslash)\|_m\leqslant c(E(u_0,f),\varepsilon)\|u^{\varepsilon,m}\|_m+k(f,m)$$

and then, easier here than Groñwall lemma, the general differential inequations theory, see [7] §V, gives us

$$\|u^{\varepsilon,m}(T,\backslash)\|_m\leqslant ae^{bT}\quad\&\quad\frac{d}{dt}\|u^{\varepsilon,m}(T,\backslash)\|_m\leqslant cae^{bT}+k=M$$

The globality-in-time of each solution $u^{\varepsilon,m}$ follows and we have then, for all m, $u^{\varepsilon,m}=u^{\varepsilon}\in H^{\infty}$

second step: the local-in-time solution:

The insertion of f in the calculus modifies the "H^m energy estimate", [6] Eq. (3.58), as follows: $\exists c_m^{(1)}>0,c_m^{(2)}>0:\forall\varepsilon>0,$

$$\frac{1}{2}\frac{d}{dt}\|u^{\varepsilon}\|_m^2+\nu\,\|J_{\varepsilon}\nabla u^{\varepsilon}\|_m^2\leqslant c_m^{(1)}|\nabla J_{\varepsilon}u^{\varepsilon}|_{L^\infty}\|u^{\varepsilon}\|_m^2+c_m^{(2)}\|u^{\varepsilon}\|_m\ (EqI.3)$$

Here, we reach the point where we have to solve the problem of the time-dependence of time T^* and, for that, momentarily to diverge from the reference proof and, hence, to give a more explicit proof of this step:
Given the Sobolev Theorem, we have: $|J_e\nabla u^{\varepsilon}|_{L^\infty}\leqslant|\nabla u^{\varepsilon}|_{L^\infty}\leqslant C\|u^{\varepsilon}\|_{m_N}\leqslant C\|u^{\varepsilon}\|_m$ for all $m\geqslant m_N=\left[{}^{N}\!/_{2}\right]+2>{}^{N}\!/_{2}+1\ \ (EqI.4)$.
It follows then from (Eq I.3):
for $m=m_N$: $\frac{d}{dt}\|u^{\varepsilon}\|_{m_N}\leqslant C.c_{m_N}^{(1)}\|u^{\varepsilon}\|_{m_N}{}^2+c_{m_N}^{(2)}$, i.e. $\frac{d}{dt}\|u^{\varepsilon}\|_{m_N}\leqslant k_N\left(\|u^{\varepsilon}\|_{m_N}^2+1\right)$ and, hence, integrating ([6] § VII): $\exists T_N=\frac{1}{k_N}\left(\frac{\pi}{2}-\arctan\|u_0\|_{m_N}\right)>0$, such that

$$\forall T<T_N,\sup_{t\leqslant T}\|u^{\varepsilon}(t,)\|_{m_N}\leqslant\tan\left(k_N.T+\arctan\left(\|u_0\|_{m_N}\right)\right)=M_T<+\infty$$

So, we obtain in (Eq I.4): $\forall m\geqslant m_N,\ |J_{\varepsilon}\nabla u^{\varepsilon}|_{L^\infty}\leqslant CM_T\ (Eq\,I.5)$
Then, similarly to [6], ref (3.59), it follows from (Eq I.3) that: $\frac{d}{dt}\|u^{\varepsilon}\|_m\leqslant c_m^{(1)}M_T\|u^{\varepsilon}\|_m+c_m^{(2)}$
Using Grönwall lemma, we obtain then:

$$\exists M'_T=M'(T,T_N,m)>0\,,\ \sup_{t\in[0,T[}\|u^{\varepsilon}(t,)\|_m\leqslant M'_T\qquad(EqI.6)$$

i.e., likewise to the reference proof, it follows that, for all $m\geqslant m_N$, the families (u^{ε}) and $\left({}^{du^{\varepsilon}}\!/_{dt}\right)$ are both uniformly bounded in $C^0([0,T],H^m)$ and $C^0([0,T],H^{m-2})$ respectively, for all $T<T_N$.
The continuation of the proof is strictly the same as the reference proof, but always taking into account that the convergence towards the solution is obtained on $[0,T_N[$ with T_N (dependent only of N), for all $m\geqslant\left[{}^{N}\!/_{2}\right]+2$.
Hence, T^* is also independent of m and the local solution belongs to

$$\bigcap_{m\geqslant m_N}\left(C^0\left([0,T^*[,H^{m,f}\right)\cap C^1\left([0,T^*[,H^{m-2,f}\right)\right)=C^1\left([0,T^*[,H^{\infty,f}\right)$$

The last results follow immediately from Energy theorem and Sobolev theorem

II. Globality-in-time of the maximal solution

Theorem 2 .1: Under the hypothesis of Theorem 1.2, the solution u is defined and smooth on $[0,+\infty[\times\mathbb{R}^N$

proof of Theorem 2.1:

Let us assume now that $T^* < +\infty$ and let $T^* < T < +\infty$. Theorem 1.1.(Eq 1.1) gives us:

$$\int_0^t |\nabla u^\varepsilon(s,)|_{L^2}^2\, ds \leqslant \int_0^t |Pf(s,)|_{L^2}\, |u^\varepsilon(s,)|_{L^2}\, ds + \frac{1}{2}|u_0|_{L^2}^2, \forall t>0$$

and then, using the energy bound $\sup_{t\geqslant 0}|u^\varepsilon(t,)|_{L^2} \leqslant E_{(u_0,f)}$ and the inequality $|Pw|_{L^2} \leqslant |w|_{L^2}$ [6], we obtain:

$$\int_0^{+\infty} |\nabla u^\varepsilon(s,)|_{L^2}^2\, ds \leqslant E_{(u_0,f)} \int_0^{+\infty} |f(s,)|_{L^2}\, ds + \frac{1}{2}|u_0|_{L^2}^2 = e_{(u_0,f)}^2 \quad (Eq2.1)$$

It follows that the sequences $(u^\varepsilon), \varepsilon>0$ and $(\partial_{x_i}u^\varepsilon), i=1,...N, \varepsilon>0$ are bounded in the Hilbert space $L^2\left(]0,T[\times\mathbb{R}^N\right)$. Hence, following the Alaoglu 's theorem, it exists $U:]0,+\infty[\times\mathbb{R}^N \to \mathbb{R}^N$ and $U_{x_i}:]0,+\infty[\times\mathbb{R}^N \to \mathbb{R}^N, i=1,...N$ such that $(u^\varepsilon), \varepsilon>0$ and $(\partial_{x_i}u^\varepsilon), i=1,...N, \varepsilon>0$ weakly converge to U and $U_{x_i}, i=1,...N$, in $L^2\left(]0,T[\times\mathbb{R}^N\right)$ Furthermore, it is then clear that $U_{x_i} = \partial_{x_i}U, i=1,...N$ i.e $\nabla U = (U_{x_i})_{i=1,...N}$ and that $U(t,) = u(t,), \forall t < T^*$ $(Eq2.2)$

We have, for all $0<t<T$, $U(t,)$ and $\nabla U(t,) \in L^2\left(\mathbb{R}^N\right)$, so $PU(t,)$ and $\widehat{\nabla U(t,)}$ belong to $L^2\left(\mathbb{R}^N\right)$ and then:

$$P\left(U(t,).\nabla U(t,)\right) = PU(t,) * \widehat{\nabla U(t,)} \in L^\infty\left(\mathbb{R}^N\right):$$

Let F_T the function defined by:
$F_T(t,) = Pf(t,) - P\left(U(t,).\nabla U(t,)\right)$ if $t\in[0,T[, = 0$ if $t\notin[0,T[$,
then it follows from (Eq1.2) and (Eq 2.1) that:

$$|F_T|_{L^\infty(\mathbb{R}^{N+1})} = \left|PU(t) * \widehat{\nabla U(t,)}\right|_{L^\infty(\mathbb{R}^{N+1})} \leqslant |PU(t)|_{L^2([0,T]\times\mathbb{R}^N)} \cdot \left|\widehat{\nabla U}\right|_{L^2([0,T]\times\mathbb{R}^N)}$$
$$\leqslant T. \sup_{t\in[0,T]} |U(t)|_{L^2(\mathbb{R}^N)} \cdot |\nabla U|_{L^2([0,T]\times\mathbb{R}^N)} \leqslant T.E(u_0 f).e(u_0 f)$$

Hence, F_T belongs to $L^\infty\left(]0,T[\times\mathbb{R}^N\right)$ and, furthermore, for all $t<T^*$, since then $U(t,) = u(t,)$, we have $F_T(t,) = Pf(t,) - P\left(u(t,).\nabla u(t,)\right)$.
Let now G the Gaussian kernel:

$$G(t,x) = \frac{1}{(4\pi\nu t)^{N/2}}\, e^{-|x|^2/4\nu t} \text{ if } t>0,\ x\in\mathbb{R}^N, = 0 \text{ if } t\leqslant 0.$$

Following [1], [3], if $\overline{u_1} = G *_{(t,x)} (F_T)$ and $\overline{u_2} = G *_{(x)} (u_0 - \overline{u}_1 (0,))$, then $\overline{u} = \overline{u}_1 + \overline{u}_2$ is solution to the Heat equation:

$$\left| \begin{array}{l} \partial_t w - \nu \, \Delta w = F_T(t, \,) \; \forall t > 0, x \in \mathbb{R}^N \\ w\,(0,) = u_0 \end{array} \right. \quad (2.3)$$

As, furthermore, all derivatives of u_0 are both smooth and bounded, it follows from properties of the Gaussian kernel that: (2.4) $\overline{u}$ is smooth on $[0, T[\times \mathbb{R}^N$ [1].[3]

On $[0, T^*[$, we have $U(t, \,) = u(t, \,)$ and then this equation is written

$$\left| \begin{array}{l} \partial_t w - \nu \, \Delta w = Pf(t, \,) - P\,(u\,(t,)\,.\nabla u\,(t,)) \;\; \forall 0 < t < T^*, x \in \mathbb{R}^N \\ w\,(0,) = u_0 \end{array} \right.$$

Hence, u, maximal solution to the Navier-Stokes equation, and $\overline{u}$ are two solutions to the restriction to $]0, T^*[\times \mathbb{R}^N$ of the equation (Eq 2.3).

Let now $0 < T' < T^* < T$, $k > 0$ and $\Omega_k = \left\{ x \in \mathbb{R}^N, \, |x| < k \right\}$.
It follows then from Theorem 1.2 and (2.4) respectively that, for all $t \in]0, T'[$, $u(t, \,)$ and $\overline{u}(t, \,)$ belong to ${H^1}_0(\Omega_k) = H^1(\Omega_k)$ since Ω_k is a bounded open set with a piecewise smooth boundary, [5].

We have then, for all $t \in]0, T'[$:
$(u - \overline{u})\,(t,) \in H_0^1\,(\Omega_k)$, $\partial_t\,(u - \overline{u})\,(t, x) - \nu\,\Delta\,(u - \overline{u})\,(t, x) = 0$, $\forall x \in \Omega_k$ and $(u - \overline{u})\,(0,) = 0$
It follows then:

$$\int_0^t \int_{\Omega_k} (\partial_t\,(u - \overline{u}) - \nu\,\Delta\,(u - \overline{u}))dx\,ds = \; ^1\!/_2\,|(u - \overline{u})\,(t,)|^2_{L^2(\Omega_k)} + \nu \int_0^t \; |\nabla\,(u - \overline{u})\,(t,)|^2_{L^2(\Omega_k)}\,dt = 0$$

and hence: $u\,(t, x) = \overline{u}\,(t, x)\,, \;\; \forall t < T', \forall x \in \Omega_k, \, \forall 0 < T' < T^*, \forall k > 0$
We have hence: $u\,(t, x) = \overline{u}\,(t, x)\,, \;\; \forall t \in [0, T^*[, \forall x \in \mathbb{R}^N$ and from that we deduce then:

(i) u belongs to $C^\infty\left([0, T^*[\times \mathbb{R}^N\right)$

(ii) u can be smoothly extended to $[0, T^*]$ setting $u(T^*, \,) = \overline{u}(T^*, \,)$ whih contradicts the maximality of T^* and hence: $T^* = +\infty$.
The proof is completed.

III. Optimality:

In the following classical decreasing sequence of functional work-spaces, in which we have obviously inserted the space H^∞: $H^m \supset H^\infty \supset S \supset D$, S is the largest space included in H^∞ and we are going now to be interested in the following question : if the initial velocity u_0 belongs to S, does the only solution u to the Navier-Stokes equation belongs to $C^\infty([0, +\infty[, \, S^f)$?, this space being defined analogously to $C^\infty([0, +\infty[, \, H^{\infty, f})$ (S is a nuclear space).

Unfortunately, We will see that such a refinement is impossible.
For that, We will first characterize the stability domain of the Leray's projector on S (Theorem 3.2) and next deduce from it the above impossibility.
The main theorem of this part is:

Theorem 3.1: (*Optimality Theorem*)

For any initial velocity u_0 in S^f, there exists external forces f in $S(0, +\infty)$ such that the only solution u to the associated Navier-Stokes equation does not belongs to the set

$\left\{w \in C^{\infty}\left([0,+\infty[\times \mathbb{R}^N\right) / w\,(t,) \in S, \partial_t w\,(t,) \in S, \forall t \geqslant 0\right\}$ and the set of such external forces is dense in $S(0,+\infty)$.

Theorem 3.2 (*Stability domain of the Leray projector*): Let w in S, then its Leray's image Pw belongs to S if and only if all moments of its divergence are null.

Lemma 3.1: $w : \mathbb{R}^N \to \mathbb{R}^N$ belongs to S if and only if one of the following equivalent properties is verified:

(a) $x^{\alpha}\partial^{\beta}w$ belongs to L^2 for all α, β in $\mathbb{N}^N$
(b) $\partial^{\alpha}(x^{\beta}w)$ belongs to L^2 for all α, β in $\mathbb{N}^N$

Proof of lemma 3.1:
The equivalence of properties (a) and (b) follows immediately from the equivalence of the semi-norms systems $|w|^{(m)}$ and $|w|^{[m]} = \sup\limits_{\max\{|\alpha|,|\beta|\} \leqslant m} \left|\partial^{\beta}\left(x^{\alpha}w\right)\right|_{L^{\infty}}$ $(see[7])$
If $w \in S$, we have $x^{\alpha}\partial^{\beta}w \in S \subset L^2, \forall \alpha, \beta$.
Reciprocally, let us assume that w verifies properties $(a), (b)$ and let $\varphi = \widehat{w}$.
We have then: $\partial^{\alpha}\left(x^{\beta}\varphi\right) = c_{\alpha,\beta}\widehat{x^{\alpha}\partial^{\beta}w} \in L^2$, $\forall \alpha, \beta$ and hence, it follows from Sobolev's theorem that:

$$\forall k \geqslant 0, \forall m > {}^{N}\!/_{2} + k, \forall \beta,\ |\beta| \leqslant m: \ \ x^{\beta}\varphi \in H^m \subset\!> C_0^k \quad ([1])$$

i.e: $\left|x^{\beta}\varphi\right|_{C_0^k} = \sup\limits_{|\alpha| \leqslant k}\left|\partial^{\alpha}\left(X^{\beta}\varphi\right)\right|_{L^{\infty}} < c_{m,k}\left\|X^{\beta}\varphi\right\|_m, \forall \beta, |\beta| \leqslant m$ and then:

$$\sup_{|\alpha|,|\beta| \leqslant k}\left|\partial^{\alpha}\left(x^{\beta}\varphi\right)\right|_{L^{\infty}} < c_{m,k}\sup_{|\beta| \leqslant k}\left\|x^{\beta}\varphi\right\|_m = c_{m,k}\sup_{|\beta| \leqslant k}\left\|x^{\beta}\widehat{w}\right\|_m$$

$$= c'_{m,k}\sup_{|\beta| \leqslant k}\left\|\widehat{\partial^{\beta}w}\right\|_m \leqslant c''_{m,k}\sup_{|\beta| \leqslant k}\left(\sum_{|\gamma| \leqslant m}\left|x^{\gamma}\partial^{\beta}w\right|_{L^2}^2\right)^{\frac{1}{2}} < \infty$$

As k is arbitrary, we deduce that φ and then $w = \varphi^{v}$ belong to S.

Lemma 3.2:

$$\forall \alpha,\ \ \partial^{\alpha}\left(\frac{X^{\beta}X_k}{|X|^2}\right) = \frac{Q_{\alpha,\beta,k}\left(X\right)}{|X|^{2(|\alpha|+1)}},\ \text{ with }\ Q_{\alpha,\beta,k}\left(X\right) = \sum_{|\rho|=|\beta|+|\alpha|+1} q_{\rho}^{\alpha,\beta,k}X^{\rho}$$

Proof of lemma 3.2:
We will proceed by induction on $|\alpha|$:
$|\alpha| = 0$: the formula is trivially true.
Assume that, for one fixed k, the formula is correct for any α, $|\alpha| = k$.
Let then $\alpha, |\alpha| = k+1$: $\alpha = \alpha^* + \delta_j$, with $|\alpha^*| = k$ and we have:

$$\partial^{\alpha}\left(\frac{x^{\beta}x_k}{|x|^2}\right) = \partial_j\partial^{\alpha^*}\left(\frac{x^{\beta}x_k}{|x|^2}\right) = \frac{\partial_j Q_{\alpha^*,\beta,k}\left(x\right)|x|^2 - Q_{\alpha^*,\beta,k}\left(x\right).2^{|\alpha|}.2x_j}{|x|^{2(|\alpha^*|+1)+1}} = \frac{Q_{\alpha,\beta,k}\left(x\right)}{|X|^{2(|\alpha|+1)}}$$

and it is clear that: $Q_{\alpha,\beta,k}\left(x\right) = \sum\limits_{|\rho|=(|\alpha|)+|\beta|+1} q_{\rho}^{\alpha,\beta,k}X^{\rho}$

Proof of Theorem 3.2

It follows from lemma III.1 that: $Pw \in S \Leftrightarrow x^{\alpha}\partial^{\beta}Pw = x^{\alpha}P\partial^{\beta}w \in L^2, \forall\alpha,\beta$
Since $Pw = \left(\frac{x}{|x|^2}\widehat{div\, w}\right)^{\vee}$ [2], and $\partial^{\beta}Pw = P\partial^{\beta}w$ [6], we obtain then:

$$Pw \in S \Leftrightarrow x^{\alpha}P\partial^{\beta}w \in L^2 \Leftrightarrow \left(\partial^{\alpha}\left(\frac{x}{|x|^2}\widehat{div\,\partial^{\beta}w}\right)\right)^{\vee} \in L^2$$

$$\Leftrightarrow \partial^{\alpha}\left(\frac{x^{\beta}.x}{|x|^2}\widehat{div\, w}\right) \in L^2 \text{ since } \widehat{div\,\partial^{\beta}w} = \widehat{\partial^{\beta}div\, w} = (-2i\pi x)^{\beta}\,\widehat{div\, w}$$

First, we can deduce from $(II.1)$ that, for $|x|>1$: $\left|\partial^{\alpha}\left(\frac{x^{\beta}x_k}{|x|^2}\right)\widehat{div\, w}(x)\right| \leqslant \left|Q_{\alpha,\beta,k}(x)\,\widehat{div\, w}(x)\right|$
and hence:

$$1_{|x|\geqslant\delta}\partial^{\alpha}\left(\frac{x^{\beta}x_k}{|x|^2}\right)\widehat{div\, w}(x) \in L^2, \forall\delta>0$$

On the other hand, we have the following alternative $(a)/(b)$:

(a) There exists at least one moment $\int x^{\tau_0}div\, w\, dx$ of $divw$ which is not null and without loss of generality, we can then choose τ_0, such that $|\tau_0|$ is minimal.
Hence, we have then:

$$\partial^{\tau_0}\widehat{div\, w}(0) = (2i\pi)^{\tau_0}\int x^{\tau_0}div\, w\, dx = m_{\tau_0} \neq 0$$
$$\partial^{\tau}\widehat{div\, w}(0) = (2i\pi)^{\tau}\int x^{\tau}div\, w\, dx = 0,\ \forall\tau/\ |\tau| < |\tau_0|$$

Let then $(\alpha,\beta) = (\alpha,0)$, it follows from lemma 3.2 that, for $\tau_0 \leqslant \alpha$:

$$\partial^{\alpha-\tau_0}\left(\frac{x_k}{|x|^2}\right) = \frac{Q_{\alpha-\tau_0,0,k}(x)}{|x|^{2(|\alpha|-|\tau_0|+1)}},\ Q_{\alpha-\tau_0,0,k}(X) = \sum_{|\rho|=|\alpha|-|\tau_0|+1} q_{\rho}^{\alpha,0,k}x^{\rho}$$

and hence we obtain, for $|x|<\delta<<1$:

$$\partial^{\alpha-\tau_0}\left(\frac{X_k}{|X|^2}\right)\partial^{\tau_0}\widehat{div\, w} \sim \frac{1}{r^{2(|\alpha|-|\tau_0|+1)}}\sum_{|\rho|=|\alpha|-|\tau_0|+1} q_{\rho}^{\alpha,0,k}r^{|\alpha|-|\tau_0|+1}T_{\rho}(\theta)\, m_{\tau_0}$$
$$\sim P_{\tau_0}(\theta)\ r^{|\alpha|-|\tau_0|+1-2(|\alpha|-|\tau_0|+1)}$$

where $T_{\alpha}(\theta)$ is the trigonometric polynomial such that:

$$x^{\alpha} = r^{|\alpha|}T_{\alpha}(\theta),\ \ \theta\in\Omega = \left\{\theta = (\theta_1,...,\theta_{N-1}) \in \mathbb{R}^{N-1}/ -\pi/2 < \theta_1,...,\theta_{N-2}, 0<\theta_{N-1}<2\pi\right\}\ \ [7]$$

Hence, we have the integral convergence equivalence:

$$\int_{|x|<\delta}\left|\partial^{\alpha-\tau_0}\left(\frac{x_k}{|x|^2}\right)\partial^{\tau_0}\widehat{div\, w}(x)\right|^2 dx \sim \int_{0<r<\delta,\theta\in\Omega}\left(P_{\tau_0}(\theta)\ r^{|\alpha|-|\tau_0|+1-2(|\alpha|-|\tau_0|+1)}\right)^2\psi(\theta)\ r^{N-1}dr\, d\theta$$
$$= \int_{0<r<\delta} r^{N+1+2\left(|\alpha|-|\tau_0|-2(|\alpha|-|\tau_0|+1)\right)}dr\,.\int_{\theta\in\Omega}P_{\tau_0}(\theta)^2\,\psi(\theta)\,d\theta\ \ (II.2)$$

It follows then that $\partial^{\alpha-\tau_0}\left(\frac{x_k}{|x|^2}\right)\partial^{\tau_0}\widehat{div\, w}(x)$ belongs to L^2 if and only if

$$N+1+2\left(|\alpha|-|\tau_0|-2^{|\alpha|-|\tau_0|+1}\right) \geqslant 0$$

and likewise for any $\tau_0' \leqslant \alpha \,/\ \ |\tau_0'| = |\tau_0|\,,\, m_{\tau_0'} \neq 0$

Let $|\alpha|$ minimal such that $N+1+2\left(|\alpha|-|\tau_0|-2^{|\alpha|-|\tau_0|+1}\right)<0$, we obtain:
(i) For $\gamma = \tau_0' + \rho \leqslant \alpha$:

$$N+1+2\left(|\alpha|-|\tau_0'+\rho|-2^{|\alpha|-|\tau_0'+\rho|+1}\right) = N+1+2\left(|\alpha-\rho|-|\tau_0|\right)-2^{|\alpha-\rho|-|\tau_0|+1} \geqslant 0$$

We have then convergence for the integral $(II.2)$ which m_γ be null or not.
(ii) For $\gamma \leqslant \alpha,\ |\gamma| \leqslant |\tau_0|$:

$$\sum_{\gamma\leqslant\alpha\,,\,|\gamma|\leqslant|\tau_0|} \partial^{\alpha-\gamma}\left(\frac{x_k}{|x|^2}\right)\partial^{\gamma}\widehat{div\,w}\,(x) \sim \sum_{\gamma\leqslant\alpha\,,\,|\gamma|\leqslant|\tau_0|} r^{|\alpha|-|\gamma|+1-2^{|\alpha|-|\gamma|+1}+(|\tau_0|-|\gamma|)}P_\gamma\left(\theta\right)$$
$$\sim r^{|\alpha|-|\tau_9|+1-2^{|\alpha|-|\tau_0|+1}} \sum_{|t_0'|=|\tau_0|} P_{\tau_0'}\left(\theta\right)$$

since $|\alpha|-|\gamma|+1-2^{|\alpha|-|\gamma|+1}+(|\tau_0|-|\gamma|) \geqslant |\alpha|-|\tau_0|+1-2^{|\alpha|-|\tau_0|+1}$ ($a \to a-2^a$ is decreasing)
Hence:

$$\int_{|x|<\delta}\sum_{|\gamma|\leqslant|\tau_0|}\left|\partial^{\alpha-\gamma}\left(\frac{x_k}{|x|^2}\right)\partial^{\gamma}\widehat{div\,w}\,(x)\right|^2 dx \sim \int_{|x|<\delta}\left|\partial^{\alpha-\tau_0}\left(\frac{x_k}{|x|^2}\right)\partial^{\tau_0}\widehat{div\,w}\,(x)\right|^2 dx$$

It follows then from the Leibniz formula that the integral$\int_{|x|<\delta}\left|\partial^{\alpha}\left(\frac{x^\beta x_k}{|x|^2}\widehat{div\,w}\,(x)\right)\right|^2 dx$ is divergent and hence, from lemma II.1, we deduce: $\partial^{\alpha}\left(\frac{x_k}{|x|^2}\widehat{div\,w}\,(x)\right) \notin L^2$ and then $Pw \notin S$.

(b) $\int x^\alpha div\,w\,dx\ = 0,\ \forall\alpha$:
We have then $\partial^\alpha\widehat{div\,w}\,(0)=0,\ \forall\alpha$ and hence $\forall\alpha,\ \partial^\rho\left(\partial^\alpha\widehat{div\,w}\right)(0)=0,\ \forall\rho.$
It follows: $\partial^\alpha\widehat{div\,w}\,(x) = o\left(|x|^m\right),\forall\alpha\in\mathbb{N}^N,\forall m\geqslant 0$ and hence, we obtain by the Leibniz formula:

$$\partial^{\alpha}\left(\frac{X^\beta.X}{|X|^2}\widehat{div\,w}\right)_k(x) = \sum_{\rho\leqslant\alpha} c_{\alpha,\rho}\partial^\rho\left(\frac{x^\beta.x_k}{|x|^2}\right)\partial^\alpha\widehat{div\,w}\,(x)$$
$$= \sum_{\rho\leqslant\alpha} c_{\alpha,\rho}\frac{Q_{\rho,\beta,k}\left(x\right)}{|x|^{2(|\alpha|+1)}}\ \partial^\alpha\widehat{div\,w}\,(x)\ = o\left(|x|^m\right),\forall m$$

It follows then that $\partial^\alpha\left(\frac{X^\beta.X}{|X|^2}\widehat{div\,w}\right)\in L^2,\forall\alpha,\beta$ and lemma II.2.1 gives us: $Pw\in S$ which achieves the proof of theorem III.2

We have then the following immediate consequences:

Consequences 3.1: The Stability domain is a closed and meagre ([4]) strict vector subspace of S which is stable by derivation and such as:

$$\forall v\in st\left(P\right),\ \forall w\in S,\ \ v*w\in st\left(P\right)$$

We can now prove the optimality theorem 3.1 :

Let $(u_0, f) \in S^f \times S(0,+\infty)$ and let us assume that the solution u to the associated Navier-Stokes equation belongs to $\left\{w \in C^\infty\left([0,+\infty[\times \mathbb{R}^N\right) / w(t,) \in S, \partial_t w(t,) \in S, \forall t \geqslant 0\right\}$. We have then: $P(u.\nabla u - f)(t,) = \nu \Delta u(t,) - \partial_t u(t,) \in S$ and, in particular for $t = 0$:

$$P(u_0.\nabla u_0 - f(0,)) \in S$$

It follows then from Theorem 3.2 that

$$\int x^\alpha div\ (u_0.\nabla u_0 - f(0,))\, dx = 0, \forall \alpha$$

and this non-trivial condition contradicts the assumed independence between the initial velocity and the external force and, so, proves the first assertion.

Let us assume that u is solution in S in the sens given in the above theorem for the constraint (u_0, f) and then that: $\int x^\alpha div\ (u_0.\nabla u_0 - f(0,))\, dx = 0, \forall \alpha$.
Let then for example $g \in S(0,+\infty)$ defined by $g(t,x) = e^{-t-|x|^2}$, $\varepsilon > 0$, $f_\varepsilon = f + \varepsilon g$ and $\alpha = (1, 0, ...0)$:

$$\int x^\alpha div\ (u_0.\nabla u_0 - f_\varepsilon(0,))\ dx\ = \underbrace{\int x^\alpha div\,(u_0.\nabla u_0 f(0,))\, dx}_{=0} - \varepsilon \int x_1 div\, g(0,)\, dx \quad \neq 0$$

Hence, for all $\varepsilon > 0$, there is no solution in S for the constraint (u_0, f_ε) and, furthermore,

$$\lim_{\varepsilon \to 0} (u_0, f_\varepsilon) = (u_0, f) \text{ in } S^f \times S(0,+\infty)$$

which proves the density.

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